Assignment # 9	Due on January 8th in class

1. In an inner product space M, suppose that there are $x, y \in M$, such that

$$|\langle x, y\rangle| = \|x\| \cdot \|y\| \ .$$

Prove that x and y are linear dependent. In other words, prove that there exists $\lambda_1, \lambda_2 \in \mathbb{C}$, such that $\lambda_1 x + \lambda_2 y = 0$ and at least one of λ_1 and λ_2 is not zero.

Proof. In fact, this is part of the Cauchy-Schwarz inequality (also called the Cauchy-Bunyakovsky-Schwarz inequality). It is about the condition when this inequality becomes an equality.

If $y = 0$, it is easy to see that $|\langle x, y\rangle| = \|x\| \cdot \|y\|$. In this case, we do have that $0 \cdot x + 1 \cdot y = 0$.

Assume that $y \neq 0$, according to the proof of the Cauchy-Schwarz inequality, by setting $\bar{\lambda} = -\dfrac{\langle y, x\rangle}{\langle y, y\rangle}$, we have

$$\begin{aligned} 0 &\le \langle x + \lambda y, x + \lambda y\rangle \\ &= \langle x, x\rangle + \bar{\lambda}\langle x, y\rangle + \lambda\langle y, x\rangle + \lambda\bar{\lambda}\langle y, y\rangle \\ &= \langle x, x\rangle - \frac{\langle y, x\rangle}{\langle y, y} \cdot \langle x, y\rangle \\ &= \frac{\langle x, x\rangle\langle y, y\rangle - |\langle x, y\rangle|^2}{\langle y, y\rangle} \\ &= \frac{\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2}{\langle y, y\rangle}. \end{aligned}$$

It follows that, in the case $y \neq 0$, if $|\langle x, y\rangle| = \|x\| \cdot \|y\|$, then

$$\langle x + \lambda y, x + \lambda y\rangle = 0,$$

thus $x + \lambda y = 0$.

Combing both cases of $y = 0$ and $y \neq 0$, we finished the proof. □

2. Let M be a Hilbert space, and let $\{e_1, e_2, \cdots\} \subset M$, satisfying $\|e_i\| = 1$, $\forall\, i \in \mathbb{N}_{\ge 1}$, and $\langle e_i, e_j\rangle = \delta_{ij}$, where δ is the Kronecker delta defined as

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

a) [**Bessel's Inequality**] Prove that for any $x \in M$, we have

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2 .$$

Hint: Just need to show that for any $n \in \mathbb{N}_{\geq 1}$, $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$.

b) Assume $\{e_1, e_2, \cdots\}^{\perp} = \{0\}$. Prove that for all $x \in M$, we have

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2 .$$

Hint: Let $x_n = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. Show that $x_1, x_2, \cdots$ is a Cauchy sequence. As M is complete (because M is a Hilbert space), $y = \lim_{n\to\infty} x_n$ is in H. Easy to check that $\|y\|^2 = \lim_{n\to\infty} \|x_n\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$. It only remains to show that $x = y$. In other words, $\lim_{n\to\infty} \|x - x_n\| = 0$.

Proof. a) Just note that

$$0 \leq \left\langle x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i, x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\rangle .$$

Using the orthogonality of those e_i and the proof follows from calculations.

b) According to the hint, we just need to show that $x = y$. Note that $\{e_1, e_2, \cdots\}^{\perp} = \{0\}$. If we can show that $\langle x - y, e_i \rangle = 0$ for all $i \in \mathbb{N}_{\geq 1}$, then we can get $x = y$.

In fact,

$$\begin{aligned}
\langle x - y, e_i \rangle &= \langle x, e_i \rangle - \langle y, e_i \rangle \\
&= \langle x, e_i \rangle - \left\langle \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j, e_i \right\rangle \\
&= \langle x, e_i \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_j, e_i \rangle \\
&= \langle x, e_i \rangle - \langle x, e_i \rangle \\
&= 0.
\end{aligned}$$

Done!

□